CH3 CHIRAL FERMIONS

→ The SM is a chiral theory: the fermionic building blocks are Weyl fermions or chiral fermions.

→ Recall that $(i\partial\!\!\!/-m)\psi=0$ and $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$
We define $\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. One finds $(\gamma^5)^2=\mathbb{1}_4$ and $\mathrm{Tr}\{\gamma^5\}=0$
In the Weyl basis, $\gamma^\mu=\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}$ and $\gamma^5=\begin{pmatrix}\mathbb{1}_2 & 0\\ 0 & \mathbb{1}_2\end{pmatrix}$

→ Dirac spinors transform under $SO(1,3)$ as:
$$\psi'(x)=\exp\{-\tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\}\psi(\Lambda x)\equiv\Lambda_{1/2}\psi(\Lambda_1 x)$$
where the following 6 matrices $S^{\mu\nu}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]$ are the gen. of $SO(1,3)$.

→ ex.: $x^\mu\mapsto x'^\mu=\Lambda^\mu{}_\nu x^\nu$. For instance, $\begin{pmatrix}t'\\x'\\y'\\z'\end{pmatrix}=\begin{bmatrix}\mathrm{ch}\theta & \mathrm{sh}\theta & 0 & 0\\ \mathrm{sh}\theta & \mathrm{ch}\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}t\\x\\y\\z\end{pmatrix}$
$\Leftrightarrow \begin{cases}t'=\mathrm{ch}\,\theta.t+\mathrm{sh}\,\theta.t\\ x'=\mathrm{sh}\theta.t+\mathrm{ch}\theta.t\end{cases}$
The preserved interval is $t^2-x^2=t'^2-x'^2$
In term of velocity v: $\begin{cases}t'=\dfrac{1}{\sqrt{1-v^2}}t+\dfrac{v}{\sqrt{1-v^2}}x=\gamma t+\gamma\beta x\\[2ex] x'=\dfrac{v}{\sqrt{1-v^2}}+\dfrac{1}{\sqrt{1-v^2}}x=\gamma\beta t+\gamma x\end{cases}$

→ $\Lambda^\mu{}_\nu$ is a rep of Lorentz group:
$x^\mu\mapsto x'^\mu=\Lambda^\mu{}_\nu x^\mu$ and $x_\mu\mapsto x'_\mu=(\Lambda^{-1})_\mu{}^\nu x_\nu$

→ For $\psi=\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$, same rotations S^{ij} but $\neq$ boosts S^{0i}
Recall $S^{ij}=\frac{1}{2}\varepsilon^{ijk}\begin{bmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{bmatrix}$ and $S^{0i}=\frac{i}{2}\begin{bmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{bmatrix}$
$\begin{cases}\chi_L'=\exp\{-\omega_{0i}\sigma^i\}\chi_L=e^{-\frac{\eta_i}{2}\sigma^i}\chi_L\\ \chi_R'=\exp\{+\omega_{0i}\sigma^i\}\chi_R=e^{\frac{\eta_i}{2}\sigma_i}\chi_R\end{cases}$ $\Big\{\eta=\mathrm{arctanh}(v)$ the rapidity
↳ Possible scalars are $\chi_L^\dagger\chi_R$ and $\chi_R^\dagger\chi_L$, so that a mass term can be $\mathcal{L}\supset y\langle\phi\rangle(\chi_R^\dagger\chi_L+h.c.)$

→ In $d=2=1+1$, $\begin{cases}\psi\text{ has 2 components}\\ \gamma^\mu\text{ is }2\times2\end{cases}$

→ Notice $\bar\psi\gamma^\mu\psi=\psi^\dagger\gamma_0\gamma^\mu\psi=(\chi_L^\dagger\ \chi_R^\dagger)\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\gamma^\mu\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}$
$=(\chi_R^\dagger\ \chi_L^\dagger)\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix}=\chi_L^\dagger\bar\sigma^\mu\chi_L+\chi_R^\dagger\sigma^\mu\chi_R$

⊙ Currents:

→ The fermionic current is $J^\mu = \bar{\psi}\gamma^\mu\psi$, a 4-vector.

It's composed of $\begin{cases} J_L^\mu \equiv \chi_L^\dagger \bar{\sigma}^\mu \chi_L \\ J_R^\mu \equiv \chi_R^\dagger \sigma^\mu \chi_R \end{cases}$

Boost along x_1: $\chi_L' = e^{-\eta/2\sigma^1}\chi_L$

$\chi_L'^\dagger = \chi_L^\dagger e^{-\eta/2\sigma^1}$ Then the L-current transforms as

$J_L^0 : \chi_L^\dagger \chi_L \mapsto \chi_L^\dagger e^{-\eta\sigma^1}\chi_L \simeq \chi_L^\dagger(1-\eta\sigma^1)\chi_L$

$= \chi_L^\dagger\chi_L - \eta\,\chi_L^\dagger\sigma^1\chi_L$

Now, $x'^0 = \mathrm{ch}\,\eta\, x^0 + \mathrm{sh}\,\eta\, x^1 \sim x^0 \pm \eta x^1$

In fact, $e^{-\eta\sigma_1} = \mathrm{ch}\,\eta - \sigma^1\,\mathrm{sh}\,\eta$

! $J_L^1 : -\chi_L^{\dagger\prime}\sigma^1\chi_L' = -\chi_L e^{-\eta\sigma_1}\sigma^1\chi_L = -\mathrm{ch}\,\eta\,\chi_L^\dagger\sigma_1\chi_L + \mathrm{sh}\,\eta\,\chi_L^\dagger\chi_L$

In general: $\chi_L \mapsto e^{-\vec{\eta}/2\cdot\vec{\sigma}}\chi_L$

$\chi_R \mapsto e^{+\vec{\eta}/2\cdot\vec{\sigma}}\chi_R$

DEF | We define the projectors as $P_L = \dfrac{1-\gamma_5}{2}$ and $P_R = \dfrac{1+\gamma_5}{2}$

PROP | $\gamma_5\psi_L = -\psi_L$ and $\gamma_5\psi_R = +\psi_R$

→ the chirality (being L or R) and the helicity $\hat{h} \equiv \dfrac{\vec{p}}{|\vec{p}|}\cdot\vec{S}$

where $\vec{S} \equiv \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$ in the Weyl basis can be linked.

→ Consider massless Weyl spinors: $\psi_L = \begin{pmatrix}\chi_L \\ 0\end{pmatrix}$ and $\psi_R = \begin{pmatrix}0 \\ \chi_R\end{pmatrix}$. Then,

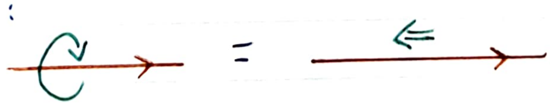
$\hat{h}\psi = \frac{\gamma_5}{2}\psi$ Indeed, for $m=0$, $i\not{\partial}\psi = 0 \Leftrightarrow \not{p}\psi = 0$

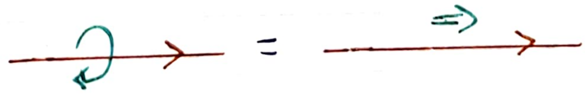
Now, $P_L(\not{p}\psi) = \not{p}P_R\psi = (E\gamma^0 - \vec{p}\cdot\vec{\gamma})P_R\psi = 0$

$\Leftrightarrow (|\vec{p}|\gamma^0 - \vec{p}\cdot\vec{\gamma})(1+\gamma^5)/2\cdot\psi = 0$

Also, $\begin{pmatrix} 0 & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & 0 \end{pmatrix}\begin{pmatrix}\chi_L \\ \chi_R\end{pmatrix} = 0 \Rightarrow \begin{cases} E\chi_R - \vec{p}\cdot\vec{\sigma}\chi_R = 0 \\ E\chi_L + \vec{p}\cdot\vec{\sigma}\chi_L = 0 \end{cases} \Leftrightarrow \begin{cases} \hat{h}\chi_R = +\chi_R/2 \\ \hat{h}\chi_L = -\chi_L/2 \end{cases}$

→ Graphically:

Left  antiparticle

Right  particle

3

⊙ Charge conjugation :

→ Naively, charge conjugation exchanges particles with antiparticles
so $e^{-ip\cdot x} \mapsto e^{ipx}$
It is not a symmetry of the SM because of the distinction between χ_L and χ_R.

→ Notice that $(i\partial\!\!\!/-m)\psi=0 \Rightarrow (i\partial\!\!\!/^*-m)\gamma^2\psi^*=0$ (since $\gamma^2\gamma^{\mu*}\gamma^2=\gamma^\mu$)

DEF | The charge conjugaison ψ^c is given by
$$\psi^c \equiv i\gamma^2\psi^* \qquad // \ \phi^c=\phi^* \text{ for scalar}$$

→ Starting from $i\gamma^\mu D_\mu\psi = m\psi \Leftrightarrow i\gamma^\mu(\partial_\mu - ieA_\mu)\psi = m\psi$: electron
$i\gamma^\mu(\partial_\mu+ieA_\mu)\psi^c = m\psi^c$: positron

Indeed, $\circledast^* \Leftrightarrow -i\gamma^{\mu*}(\partial_\mu+ieA_\mu)\psi^* = m\psi^*$
and $(\gamma^2)^2=-\mathbb{1}$, and $\sigma^2\sigma^{i*}\sigma^2=-\sigma^i \Leftrightarrow \gamma^2\gamma^{\mu*}\gamma^2=\gamma^\mu$

PROP | We can write $\psi^c = C\bar{\psi}^t$ with $C\equiv i\gamma^2\gamma^0$
↳ $C^\dagger=-C$

→ Notice $\psi^c = i\gamma^2\begin{bmatrix}\chi_L^*\\ \chi_R^*\end{bmatrix} = \begin{pmatrix}0 & i\sigma^2\\ -i\sigma^2 & 0\end{pmatrix}\begin{pmatrix}\chi_L^*\\ \chi_R^*\end{pmatrix} = \begin{pmatrix}i\sigma^2\chi_R^*\\ -i\sigma^2\chi_L^*\end{pmatrix} \equiv \begin{pmatrix}\chi_R^c\\ \chi_L^c\end{pmatrix}$

If we have a theory with $\chi_L \neq \chi_R$, C is not a good symmetry.

DEF | The parity transformation P is such that
$$P\psi(\vec{x},t) \equiv \gamma^0\psi(-\vec{x},t) \qquad // \ \phi(\vec{x},t)\mapsto\phi(-\vec{x},t)$$

→ In a theory with only χ_L, we can make the transformation
$$\chi_L \mapsto CP\chi_L \equiv -i\sigma^2\chi_L^*$$
Equivalently, $CP\chi_R = +i\sigma^2\chi_R^*$
↳ Only CP brings back a state of the theory with a chiral content.

→ Under a boost,
$$\chi_L^c \mapsto (e^{-\vec{\eta}\cdot\vec{\sigma}/2}\chi_L)^c = -i\sigma^2(e^{-\vec{\eta}\cdot\vec{\sigma}/2}\chi_L)^*$$
$$= -ie^{\vec{\eta}\cdot\vec{\sigma}/2}\sigma_2\chi_L^* = e^{+\vec{\eta}\cdot\vec{\sigma}/2}\chi_L^c$$
$\left\{ \begin{array}{l} \sigma^2\sigma^{i*}=-\sigma^i\sigma^2 \\ \text{and } \sigma^2\sigma^i\sigma^2=-\sigma^i \end{array}\right.$

PROP | χ_L^c transforms as a R-field, and χ_R^c as a L-field.

⊙ Majorana mass:

→ To write a Dirac mass, one needs 4 dof: χ_L and χ_R.
For a Majorana mass, only 2:

$$m_M \left(\chi_L^{c\,\dagger} \chi_L + \chi_L^{\dagger} \chi_L^{c}\right) \quad \text{or} \quad (L \leftrightarrow R).$$

→ For a neutral particle, we do not impose that terms are invariant under a phase transformation.
↳ Majorana mass are possible for neutrinos.

→ We can think of mass as self-interactions
At high energy, we take $m \to 0$, so that the amplitude of transitioning from L to R goes as $\propto m/E$